Mid Semestral Examination
Maximum you can score is 60
Duration: 180 minutes

(1) Let $\mathfrak{S}$ be a σ-algebra of subsets of Ω. Show that it has uncountably many elements provided it is not finite. **[12]**

(2) Let $X : \mathbb{R} \to \mathbb{R}$ be a monotone function show that X is Borel measurable. **[10]**

(3) Show that a distribution function has countably many discontinuities. **[8]**

(4) Let $(\Omega, \mathfrak{S}, P)$ be a probability space. Show that a collection of random variables $\{X_\alpha : \alpha \in A\}$ is uniformly integrable iff
 (a) $\sup_{\alpha \in A} \int |X_\alpha| dP < \infty$.
 (b) $\forall \epsilon > 0, \exists \delta > 0$ such that $\sup_{\alpha \in A} \int_B |X_\alpha| dP < \epsilon$, provided $P(B) < \delta$.
[6+6=12]

(5) Let $\{E_n\}$ be events and $\{I_n\}$ be their indicators. Prove the inequality

$$P(\cup_{k=1}^n E_k) \geq \left[\mathbb{E}\left(\sum_{k=1}^n I_k\right)\right]^2 \Bigg/ \mathbb{E}\left[\left(\sum_{k=1}^n I_k\right)^2\right].$$

[6]

(6) Let $(\Omega, \mathfrak{S}, P)$ be a probability space and X is an integrable non-negative random variable such that $\mathbb{E}^P(X) = 1$.
 (a) Show that $Q : \mathfrak{S} \to \mathbb{R}$ given by $Q(B) = \mathbb{E}^P(X 1_B)$ is a probability measure.
 (b) For a non-negative random variable Y show that $\mathbb{E}^Q(Y) = \mathbb{E}^P(XY)$.
 (c) Finally show that a random variable Y is integrable with respect Q iff XY is integrable with respect P and in that case $\mathbb{E}^Q(Y) = \mathbb{E}^P(XY)$.
[3+5+8=16]

(7) Cleanliness **[2]**